On the absence of infinite AB percolation clusters in bipartite graphs

1987 J. Phys. A: Math. Gen. 20 2527

(http://iopscience.iop.org/0305-4470/20/9/036)

J. Phys. A: Math. Gen. 20 (1987) 2527-2531. Printed in the UK

On the absence of infinite AB percolation clusters in bipartite graphs

Martin J Appel and John C Wierman

Department of Mathematical Sciences, The Johns Hopkins University, Baltimore, MD 21218 USA

Received 21 July 1986

Abstract. We give a rigorous proof that infinite AB percolation does not occur for any parameter value in a certain class of bipartite planar graphs. In particular, infinite AB percolation cannot occur on the square lattice.

1. Introduction

We consider a variant of the percolation model which is motivated by chemical bonding considerations. In this model, there are two types of atoms, A and B, which occupy the sites of an infinite lattice graph G, with probabilities p and $1-p$, respectively. Unlike atoms which are connected by an edge of G are bonded together, while like atoms which are connected do not bond. The object of study is the size (and other characteristics) of the clusters of atoms bonded together. In particular, one wishes to determine whether infinite bonded clusters exist for some parameter values, or if all bonded clusters are finite for all values of $p \in [0, 1]$.

The model has been studied previously by Halley (1983), who named it 'AB percolation', and by Sevšek *et al* (1983), who called it 'antipercolation'. Halley gave a plausibility argument to show that all bonded clusters are finite for all values of $p \in [0, 1]$ if the underlying graph G is bipartite and has a site percolation critical probability strictly greater than $\frac{1}{2}$. The argument is not a mathematically rigorous proof, however. We provide a proof that all bonded clusters are finite on a large class of bipartite graphs and describe a variety of graphs included in this class. In addition, by a different argument, we show that there are no infinite bonded clusters on the square lattice for any value of p. This result relies on a special relationship with the asymmetric bond percolation model on the square lattice. The square lattice is bipartite, and its site percolation critical probability is strictly greater than $\frac{1}{2}$ (Toth 1985), but our previously mentioned result does not apply.

In fact, it has not been rigorously proven that infinite AB percolation clusters exist with positive probability for some value of p on any infinite two-dimensional lattice. Monte Carlo evidence (Mai and Halley 1980) suggests that infinite clusters occur for $p \in [0.2145, 0.7855]$ on the triangular lattice. We provide a rigorous proof that infinite AB clusters exist for an interval of values of p on the triangular lattice in the following paper (Wierman and Appel 1987).

Definitions and notation are provided in § 2. Fundamental results from classical percolation are described in § 3. Halley's plausibility argument, and our rigorous proof for a class of graphs, appears in § 4. Examples are discussed in § 5. The result for the square lattice is proved in § 6.

2. Definitions

A graph G consists of a countable set $V(G)$ of vertices and a countable set $E(G)$ of pairs of vertices, called edges. A graph G is *bipartite* if there exists a partition of $V(G)$ into two sets V_1 and V_2 such that every edge in $E(G)$ has one endpoint in V_1 and one endpoint in V_2. The sets V_1 and V_2 are called a *bipartition.* Note that any path on a bipartite graph passes through vertices of V_1 and V_2 alternately.

An assignment of a label, A or B, to each vertex of G is a *configuration* on G, i.e. a configuration is an element $w \in \{\mathrm{A}, \mathrm{B}\}^{V(G)}$, or equivalently a function w: $V(G) \to \{\mathrm{A}, \mathrm{B}\}$. The AB percolation model on G is a probability model with sample space $\{\mathrm{A}, \mathrm{B}\}^{V(G)}$ and probability measure P_p such that the labels of the vertices of G are independent random variables with probability p of labelling each vertex A.

An edge of G is an AB *bond* if the endpoints of the edge have different labels. An AB *path* is an alternating sequence of vertices and edges $v_0, e_1, v_1, \ldots, e_n, v_n$ such that e_i, $1 \leq i \leq n$, are all AB bonds. The AB *cluster* containing a vertex v, denoted W_v^{AB}, is the set of all vertices which may be reached from v through an AB path. The number of vertices in W_v^{AB} is denoted by $\# W_v^{\mathrm{AB}}$.

Define the AB *percolation probability* by

$$\theta_v^{\mathrm{AB}}(p) = P_p(\# W_v^{\mathrm{AB}} = +\infty).$$

Note that AB paths and AB clusters are unchanged if the label of every vertex is changed, but the parameter of the model is changed from p to $1-p$. Thus, $\theta_v^{\mathrm{AB}}(p) = \theta_v^{\mathrm{AB}}(1-p)$ for all $p \in [0, 1]$, so the AB percolation probability function is symmetric about $\frac{1}{2}$.

While the value of $\theta_v^{\mathrm{AB}}(p)$ may depend on the vertex v, the set of values of p for which $\theta_v^{\mathrm{AB}}(p) > 0$ is independent of the choice of vertex if G is a connected graph, as for classical percolation models. Intuitively, one expects that $\theta_v^{\mathrm{AB}}(p) > 0$ only on a single interval, but it has not been proven that there cannot be multiple intervals.

3. Classical percolation results

We will rely on results of Kesten (1982) for the classical site percolation model, which will be stated after the following definitions.

A graph G is *periodic* in $\mathbb{R}^d$ if

(a) G is embeddable in $\mathbb{R}^d$ so that the vertex and edge sets are invariant under translation by a set of basis vectors for $\mathbb{R}^d$;

(b) there exists a finite z so that the maximum degree of the vertices of G is z;

(c) every compact set of $\mathbb{R}^d$ intersects only finitely many edges of G;

(d) G is connected.

Most of the lattices commonly used in statistical physics, such as the square lattice, correspond to periodic graphs, but self-similar and tree-like pseudolattices are excluded.

A *mosaic* M is a graph embedded on $\mathbb{R}^2$ which has finite maximum degree such that

(i) any two edges of M intersect only at endpoints,

(ii) each component of $\mathbb{R}^2 \backslash M$ is bounded by a Jordan curve consisting of a finite number of edges of M.

To a great extent, (i) excludes two-dimensional lattices which have other than nearest-neighbour connections. A mosaic M is a planar graph, and each component of $\mathbb{R}^2 \backslash M$ is called a *face* of M. If F is a face of M, *close-packing* F means adding an edge between any pair of vertices on the perimeter of F which are not yet adjacent. If M is a mosaic and $\mathscr{F}$ a subset of its faces, the *matching pair* $(\mathscr{G}, \mathscr{G}^*)$ of graphs based on $(M, \mathscr{F})$ is the following pair: $\mathscr{G}$ is constructed from M by close packing all faces of $\mathscr{F}$. $\mathscr{G}^*$ is constructed from M by close packing all faces not in $\mathscr{F}$.

Kesten's fundamental result states that if G and G^* are a matching pair of periodic graphs in $\mathbb{R}^2$ with at least one axis of symmetry, then the sum of the site percolation critical probabilities of G and G^* is one. In addition, for both G and G^*, the probability of an infinite open cluster is zero at the critical probability. One consequence is that the critical probability of any fully triangulated planar periodic graph with one axis of symmetry is $\frac{1}{2}$, since the graph is self-matching.

4. Results for bipartite graphs

Halley (1983) proved the following result by a symmetry argument when $p = \frac{1}{2}$.

Lemma 4.1. If G is a bipartite graph with site percolation critical probability strictly greater than $\frac{1}{2}$, then $\theta_v^{AB}(\frac{1}{2}) = 0$.

The result is obtained by reversing the labels on one set of the bipartition of G, converting the AB problem into a classical site problem and noting that at $p = \frac{1}{2}$ the probabilities of infinite A, B and AB clusters are all equal, and thus are equal to zero.

Using this result, Halley states that it follows that the probability of an infinite AB cluster is zero for all values of p in such lattices, based on a claim that 'the probability of an infinite AB cluster is clearly largest at $p = \frac{1}{2}$.' While the claim is intuitive, no proof is offered and we have not been able to construct a rigorous proof for it.

We next offer a partial confirmation of Halley's statement on the absence of infinite AB clusters. Let G be a bipartite graph with bipartition sets V_1 and V_2. For each V_i, construct a graph $G(V_i)$ with vertex set V_i, such that vertices u and v are adjacent in $G(V_i)$ if and only if u and v are adjacent to a common vertex in G. Let p_1 and p_2 denote the site percolation critical probabilities of $G(V_1)$ and $G(V_2)$ respectively.

Theorem 4.2. If G is a bipartite graph such that

(a) $p_1 + p_2 > 1$, or

(b) $G(V_1)$ and $G(V_2)$ are each a member of a matching pair of graphs, periodic, and have one axis of symmetry, and $p_1 + p_2 = 1$,

then $\theta_v^{AB} = 0$ for all $v \in V(G)$ for all $p \in [0, 1]$.

Proof. If there exists an infinite AB cluster in G, then there is an infinite A cluster in one $G(V_i)$ and an infinite B cluster in the other $G(V_i)$. Let $A_i(B_i)$ denote the event that there is an infinite A (B, respectively) cluster in $G(V_i)$. Then, for any v,

$$\begin{aligned}\theta_v^{AB}(p) &\leq P_p[(A_1 \cap B_2) \cup (A_2 \cap B_1)] \\ &\leq P_p[(A_1 \cap B_2) + P_p[A_2 \cap B_1].\end{aligned}$$

In case (a), note that for A_1 to occur with positive probability, $p \geq p_1$, which implies that $1-p \leq 1-p_1 < p_2$ by hypothesis, so B_2 has probability zero. Hence $P_p[A_1 \cap B_2]=0$ for all p, and similarly $P_p[A_2 \cap B_1]=0$ for all p. Thus, $\theta_v^{AB}=0$ for all $p \in [0, 1]$. In case (b), for A_1 to occur with positive probability, by Kesten's results described in § 3, $p > p_1$, which implies that $1-p < p_2$, and the argument is completed as before.

5. Examples

Example 1. Let G be any periodic bipartite mosaic with one axis of symmetry and with maximum degree three. Two vertices in V_i are adjacent in $G(V_i)$ if and only if they are on the boundary of a common face, since the vertex of G adjacent to both has degree at most 3 and thus lies on the boundary of at most three faces. The corresponding edge in $G(V_i)$ can be represented as a line segment in the common face of G. Then $G(V_i)$ is a planar graph if the edges in each face of G do not intersect. However, each face of G is bounded by an even number of edges, since G is bipartite, so each face contains either a single edge of $G(V_i)$ or a circuit of edges of $G(V_i)$. In either case, no edges of $G(V_i)$ intersect, so $G(V_i)$ is planar. Thus, each $G(V_i)$ is a periodic mosaic with one axis of symmetry, satisfying the conditions in theorem 4.2(b). Each graph $G(V_i)$ has a site percolation critical probability $p_i \geq \frac{1}{2}$, since by adding edges periodically it may be made fully triangulated. Applying theorem 4.2, with probability one there are no infinite AB clusters in G for any $p \in [0, 1]$.

A special case of this fact is the hexagonal lattice. In this case, $G(V_i)$ are both triangular lattices, so $p_1+p_2=\frac{1}{2}+\frac{1}{2}=1$, so (b) applies but (a) does not.

Any graph G, obtained from a periodic mosaic M with one axis of symmetry and maximum degree 3 by placing (periodically) an odd number of additional vertices on each edge of M, is bipartite and still has maximum degree three, so the above fact applies.

Example 2. Consider the graph G obtained by placing an additional vertex on each edge of the square lattice. G is a periodic bipartite mosaic with two axes of symmetry, but has maximum degree 4. However, the $G(V_i)$ are the square lattice and the covering graph of the square lattice, so the sum of the site percolation critical probabilities is strictly greater than one. Theorem 4.2(b) shows that there is no AB percolation on G for any $p \in [0, 1]$.

Example 3. Let G be the square lattice, which is a periodic bipartite mosaic with two axes of symmetry, but has maximum degree 4. Each $G(V_i)$ is the matching graph of the square lattice, i.e. a square lattice in which every face has been close packed. In this case, it is known that $p_i < \frac{1}{2}$, so $p_1+p_2<1$ and theorem 4.2 does not apply. Nevertheless, infinite AB percolation is impossible on the square lattice, as will be shown in the following section.

6. AB percolation on the square lattice

For our study of the square lattice, we use the usual embedding in the plane with vertex set $\{v=(v(1), v(2)) \in \mathbb{Z}^2\}$, and with an edge between v and w if and only if $\|v-w\|=[(v(1)-w(1))^2+(v(2)-w(2))^2]^{1/2}=1$. The square lattice is bipartite, with bipartition $V_1=\{v \in V(G)\colon v(1)-v(2) \text{ is odd}\}$ and $V_2=\{v \in V(G)\colon v(1)-v(2) \text{ is even}\}$.

Recalling Halley's proof of lemma 4.1, reverse the labels of all vertices in V_2, and view the result as a classical percolation model in which vertices in V_1 are labelled A with probability p and vertices in V_2 are labelled A with probability $1-p$. Reversing the labels transforms an AB cluster in the original model into either an A cluster or a B cluster in the new model.

We now use a connection with the asymmetric bond percolation model on the square lattice to demonstrate that infinite A clusters (and infinite B clusters, by symmetry) cannot exist with positive probability for any value of $p \in [0, 1]$ in the new model. In the asymmetric bond percolation model on the square lattice, each horizontal edge is open with probability p and each vertical edge is open with probability q. Kesten (1982) proved that the critical surface is $p+q=1$, so that if $p+q<1$, or $p+q=1$ with $0<p<1$, then with probability one there is no infinite open cluster.

By the bond-to-site transformation, the asymmetric bond percolation model is equivalent to a site percolation model on the covering graph of the square lattice in which sites are open with probability p or q. Thus, in this model, there are no infinite open clusters if $p+q<1$, or $p+q=1$ with $0<p<1$. (Infinite open clusters exist when $p=0$, $q=1$ and $p=1$, $q=0$.)

Removing the close-packing edges from the covering graph of the square lattice cannot increase the probability of an infinite open cluster, and produces a site percolation model on the square lattice with each vertex in one bipartition set open with probability p and each vertex in the other bipartition set open with probability q. Thus there is no infinite percolation if $p+q \leqslant 1$. (When $p+q=1$ in this model, no infinite open cluster is present when $p=0$ or 1.) Since the transformation of the AB percolation model is equivalent to this model with $p+q=1$, there are no infinite A clusters (and no infinite B clusters) for any value of $p \in [0, 1]$. Consequently there are no infinite AB clusters on the square lattice for any value of $p \in [0, 1]$.

Acknowledgments

Professor J C Wierman's research is supported in part by the National Science Foundation (grant no DMS-8303238). We thank the referee for his insightful comments and suggestions.

References

Halley J W 1983 *Percolation Structures and Processes* ed G Deutscher, R Zallen and J Adler (Bristol: Adam Hilger) pp 323-51

Kesten 1982 *Percolation Theory for Mathematicians* (Basle: Birkhauser)

Mai T and Halley J W 1980 *Ordering in Two Dimensions* ed S Sinha (Amsterdam: North-Holland) pp 369-71

Sevšek F, Debierre J-M and Turban L 1983 *J. Phys. A: Math. Gen.* **16** 801-10

Toth B 1985 *Z. Wahrsch. Verw. Geb.* **69** 19-22

Wierman J C and Appel M J 1987 *J. Phys. A: Math. Gen.* **20** 2533-7